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G.M. PETERSEN

THE ALGEBRA OF BOUNDED SEQUENCES AS FACTOR SEQUENCES

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THE ALGEBRA OF BOUNDED SEQUENCES AS FACTOR SEQUENCES

by

G.M. PETERSEN

1. Let $A = (a_{mn})$ be a regular summability matrix. By $\mathcal{A}$ is denoted the set of bounded sequences limited by A and by $\mathcal{A}_0$ those bounded sequences which are limited to zero. If $x = \{x_n\}$ and $y = \{y_n\}$, the product sequence $\{x_n y_n\}$ is denoted by xy and $\{x_n + y_n\}$ by x + y. The usual norm $||x||$ of a bounded sequence x is given by $||x|| = \sup_n |x_n|$. The bounded sequence ξ is a <u>factor sequence</u> for $\mathcal{A}$ (or $A = (a_{mn})$) if $\xi x \in \mathcal{A}_0$ whenever $x \in \mathcal{A}_0$. The set of factor sequences corresponding to $\mathcal{A}$ we denote by $\mathcal{A}^*$. With the operations of addition and multiplication defined as above and under the usual norm, $\mathcal{A}^*$ is a Banach algebra, see [6]. It is our purpose in this paper to discuss those regular matrices for which $\mathcal{A}^*$ consists of the set of all bounded sequences.

Before we can come to our main purpose, we must develop some other ideas. A sequence $\{t_n\}$ is a <u>thin</u> sequence with respect to $A = (a_{m,n})$ if $t_n = 0$, $n \notin \{n_k\}$ where

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} |a_{m,n_k}| = 0. \qquad (1)$$

A matrix $A = (a_{m,n})$ is <u>positive</u> if $a_{m,n} \geq 0$, $(m,n = 1,2,...)$ and <u>finite rowed</u> if there exists an integer valued function $\mu(m) \uparrow \infty$ such that $a_{m,n} = 0$, $n \geq \mu(m)$. In [5], the following theorem is proved though stated a different way:

<u>Theorem 1</u>. If $A = (a_{m,n})$ is a positive, finite rowed and regular matrix, $\xi \in \mathcal{A} \cap \mathcal{A}^*$, then $\xi = r+t$ where r is convergent and t is thin.

If $\mathcal{A} = \mathcal{B}$, the matrices A and B are <u>b-equivalent</u>, if $\mathcal{A} \supset \mathcal{B}$ then A is <u>b-stronger</u> than B. If $A = (a_{m,n})$ is any regular matrix, then there exists a finite rowed matrix, $A' = (a'_{m,n})$ which is b-equivalent to A, see [4] page 82, in particular A' has the same thin sequences as A.

For this reason we need only consider finite rowed matrices in the sequel. The following theorem is then an immediate consequence of Theorem 1:

Theorem 2. If $A = (a_{mn})$ is a positive regular matrix such that A^* is the set of all bounded sequences, then $x \in A$ if and only if $x = r+t$ where r is convergent and t is thin.

A matrix defined by the transformations,

$$\tau_k = x_{n_k} \qquad (k = 1,2,\ldots)$$

where $\{n_k\}$ is an infinite subsequence of the natural numbers, satisfies the conditions of Theorem 2. This is also true of the matrix described by Garreau [1], see also [3]. We now prove:

Theorem 3. Let $A = (a_{m,n})$ be a regular matrix. The following two conditions are equivalent:

1) A^* is the set of all bounded sequences
2) $x \in A$ if and only if $x = r+t$ where t is convergent and t is thin.

Proof. We may in fact assume that $a_{mn} = 0$ whenever $n \leq \lambda(m)$ or $n \geq \mu(m)$, $\lambda(m) \uparrow \infty$, $\mu(m) \uparrow \infty$, see [4], page 82. Now suppose there exists $x \in A$ which is not of the form given; then in fact we can suppose $x \in A_0$. Since x is not the sum of a sequence convergent to zero and one thin with respect to A, for infinitely many m and some $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ we have

$$|x_{n_r}| > \varepsilon_1, \qquad (r = 1,2,\ldots) \qquad (2)$$

and

$$\sum_{r=1}^{\infty} |a_{m,n_r}| > \varepsilon_2 \qquad (3)$$

where it is important to note that $\{n_r\}$ may depend on m, see [4]. However, we select $\{m_k\}$ so that (3) is satisfied whenever $m = m_k$ and such that $\mu(m_k) \leq \lambda(m_{k+1})$, (k=1,2,...). Then the same sequence $\{n_r\}$ can be chosen to all rows m_k (k=1,2,...). Moreover x is not thin with respect to the matrix $B = (b_{kn})$ defined by $b_{kn} = a_{m_k n}$. The matrix B has at most one non-zero element in each column, hence we can define sgn n by sgn n = 1 if the non-zero element in the n-th column is positive, or if all elements in the column are zero; sgn n = −1 if the non-zero element is negative. Consider now the matrix $C = (c_{kn})$ defined by

$$c_{kn} = b_{kn} \operatorname{sgn} n \,/\, \sum_{n=1}^{\infty} |b_{kn}|.$$

The matrix C is regular and positive. The sequence $x \in \mathcal{C}_0$ since $\{x_n \operatorname{sgn} n\} \in \mathcal{B}_0$, i.e. $\{\operatorname{sgn} n\} \in \mathcal{B}^*$. The sequence x is not thin with respect to C since

$$\sum_{r=1}^{\infty} |c_{k,n_r}| > \varepsilon_2 \,/\, \sum_{n=1}^{\infty} |b_{k,n}|.$$

However if $y \in \mathcal{C}_0$ and ξ is any bounded sequence, $\{y_n \operatorname{sgn} n\} \in \mathcal{B}_0$ and so $\{\xi_n\, y_n \operatorname{sgn} n\} \in \mathcal{B}_0$ and this implies $\xi y \in \mathcal{C}_0$ so that $\mathcal{C}^*$ is the set of bounded sequences. Hence we would have that $\mathcal{C}^*$ is the set of bounded sequences, but $x \in \mathcal{C}_0$ is not of the form r + t where t is convergent and t is thin. This contradiction proves our theorem.

Using the matrices defined in the proof of Theorem 3, it is easy to prove the following theorem which characterizes thin sequences in terms of the summability field:

Theorem 4. If $x \in \mathcal{A}_0$, then x = r+t where r converges to zero and t is thin if and only if $\xi x \in \mathcal{A}_0$ for all bounded sequences ξ.

We now prove:

Theorem 5. If $A = (a_{m,n})$ is a regular matrix such that $x \in \mathcal{A}$ implies that x = r+t where r is convergent and t is thin, then there exists a positive regular matrix $B = (b_{mn})$ such that $\mathcal{B} \supset \mathcal{A}$ and such that A and B have the same thin sequences.

Proof. Let $B = (b_{mn})$ be defined by

$$b_{mn} = |a_{mn}| \,/\, \sum_{n=1}^{\infty} |a_{mn}| \qquad (m,n = 1,2,\ldots).$$

Then t is thin with respect to A if and only if it is thin with respect to B. This completes our proof.

We now prove the following:

Theorem 6. Let $A = (a_{mn})$ be a positive regular matrix. Then there exists a positive regular matrix $B = (b_{mn})$ such that $x \in \mathcal{B}$ if and only if x = r+t where r is convergent and t is thin with respect to A.

Proof. An auxiliary matrix $C = (c_{mn})$ is first defined; suppose the first N rows of the matrix C have been defined, where

$$N = 2^{\mu(1)} + 2^{\mu(2)} + \dots + 2^{\mu(m)} \tag{4}$$

where $a_{mn} = 0$, $n \geq \mu(m)$. Then the next $2^{\mu(m+1)}$ rows will be defined by first taking the $(m+1)$st row of A as the $N + 1$ st row of C. The next $\mu(m+1)$ rows of C will consist of the $(m+1)$st row of A with each of the different elements $a_{m+1,n}$ $1 \leq n \leq \mu(m+1) - 1$ successively replaced by a zero, the other $(m+1) - 1$ elements remaining the same. Now two elements of the row are replaced by zeros in all possible ${}_{\mu(m+1)}C_2$ ways. The next ${}_{\mu(m+1)}C_3$ rows are determined by replacing three elements by zeros. In all $2^{\mu(m+1)}$ rows are used as the elements of the $m + 1$ st row of A replaced in all possible ways by 1, 2, ..., $\mu(m+1)$ zeros. The new matrix C is certainly not regular. However, if N is defined as in (4), it is clear that for $N \leq r \leq N+2^{\mu(m-1)}-1$

$$\sum |c_{r,n_k}| = \sum c_{r,n_k} \leq \sum a_{m,n_k} = \sum |a_{mn_k}| \tag{5}$$

where $\{n_k\}$ is any subsequence of the natural numbers. Consequently a sequence which is thin with respect to A is thin with respect to C. If x cannot be expressed as $r + t$, where r converges to zero and t is thin, then (3) is satisfied and we may even assume

$$x_{n_k} > \varepsilon_1 \qquad (k = 1,2,\dots)$$

or

$$x_{n_k} < -\varepsilon_1 \qquad (k = 1,2,\dots) \tag{6}$$

and for some row ν

$$N+1 \leq \nu \leq N+2^{\mu(m+1)} \tag{7}$$

we have either

$$\sum c_{\nu n}\, x_n = \sum c_{\nu n_k}\, x_{n_k} = \sum a_{mn_k}\, x_{n_k} \geq \varepsilon_1\, \varepsilon_2 \tag{8}$$

or

$$\sum c_{\nu n}\, x_n < -\, \varepsilon_1\, \varepsilon_2 \tag{9}$$

so that x is not limited to zero by C. Hence the only $x \in A_0$ limited to zero by C are the thin sequences. For every ν satisfying (7), let

$$\sum c_{\nu n} = \eta_\nu$$

and define $B = (b_{\nu n})$ by

$$b_{\nu n} = (1-\eta_\nu)\, a_{mn} + c_{\nu n}\ ;$$

we can assume that $\sum a_{mn} = 1$, $(m=1,2,\dots)$ so that $\eta_1 = 0$ and $0 \leq \eta_\nu \leq 1$, $(N+1 \leq \nu \leq N+2^{\mu(m+1)})$. It is clear from the construction that B is regular; since $b_{N+1,n} = a_{m,n}$ $(n=1,2,\dots)$ it follows that $B_0 \subset A_0$. If $x \in A_0$ but is not limited to zero by C, it is clear from (8) and (9) that

$$\limsup |\sum c_{\nu n}\, x_n| \geq \varepsilon_1\, \varepsilon_2$$

while

$$\lim \sum c_{N+1,n}\, x_n = 0.$$

Hence $x \in B_0$ if and only if $x \in A_0$ and is limited to zero by C; that is $x = r+t$ where t is thin and r converges to zero. On the other hand, it is clear from (5) that all sequences thin with respect to A are limited to zero by B. This completes the proof.

Combining Theorems 3, 5 and 6, we have:

Theorem 7. Let $A = (a_{m,n})$ be a regular matrix. If A^* is the set of bounded sequences, A is b-equivalent to a positive matrix.

The construction used in Theorem 5 is similar to that of Garreau [1] with respect to the (C,1) matrix. However, Garreau's construction could be slightly more economical owing to the special properties of the (C,1) matrix. The construction of Theorem 5 shows that there are many of these matrices since any regular matrix may be used as a basis for the construction. Of course a positive matrix with the bounded sequences as the algebra already will produce a second matrix b-equivalent to the first.

2. The matrix $A = (a_{m,n})$ is <u>b-multiplicative</u> if whenever $x \in A_0$ and $y \in A_0$ then $xy \in A_0$, see [2]. If $A = (a_{mn})$ is b-multiplicative and $x \in A_0$ is not the sum of a thin sequence and one convergent to zero, construct the matrix C exactly as in the proof of Theorem 3. Let $y = \{x_n \operatorname{sgn} n\}$, then C limits y to zero and also y^k $(k=1,2,\ldots)$. Then since y is not thin with respect to C, we arrive at a contradiction as in [5]. Hence $A = (a_{mn})$ is b-multiplicative if and only if all members of A are of the form r + t where r is convergent and t is thin. This leads to the following result:

<u>Theorem 8</u>. The regular matrix $A = (a_{mn})$ is b-multiplicative if and only if the set of factor sequences is the set of bounded sequences.

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